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THE EFFECT OF AXIALLY VARYING AND UNSYMMETRICAL BOUNDARY CONDITIONS ON HEAT TRANSFER WITH TURBULENT FLOW BETWEEN PARALLEL PLATES

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Abstract—The solution is presented to the heat-transfer problem with turbulent flow between parallel plates with heating on one side only. The configuration has some practical significance when considered as a limiting case of the annulus with a heated core. The velocity and eddy diffusivity variations due to Deissler are used but are modified in the central region of the passage by assuming a constant eddy diffusivity. In solving the energy equation the eddy diffusivities for heat and momentum are taken equal. The eigenvalues and functions necessary to calculate the variation of Nusselt number with distance along the passages are tabulated for three Reynolds and three Prandtl numbers.

From the two basic solutions of uniform temperature and uniform heat flux the analysis is extended to certain axial variations which are of practical interest and finally, by using superposition, the result is obtained for unequal uniform heat fluxes on each side of the passage.

NOMENCLATURE

A, ratio of maximum heat flux due to sine component to uniform heat flux when components act;
C, constants in the solution of the Sturm–Liouville equation;
d, mean diameter defined as $4 \times$ area/perimeter $= 4y_o$;
G, fully developed temperature profile (having a bulk mean value of zero);
h, heat-transfer coefficient based on the differences of wall to bulk mean temperature;
k, fluid thermal conductivity;
q, heat flux at wall;
t, temperature;
T, dimensionless temperature $\dfrac{t - t_i}{t_e - t_i}$;
u, velocity;
u^+, dimensionless velocity $u/\sqrt{(\tau_w/\rho)}$;
x, distance from the entrance;
x^+, dimensionless distance x/d;
y, distance from the wall;
y_0, half-width of passage;
y^+, dimensionless distance from the wall $\dfrac{y\sqrt{(\tau_w/\rho)}}{\nu}$;
Y, eigenfunction in the solution;
Nu, Nusselt number hd/k;
Re, Reynolds number $u_m d/\nu$;
Pr, Prandtl number, ν/α.

Greek symbols

α, thermal molecular diffusivity;
ϵ, eddy diffusivity;
λ, eigenvalue;
θ, dimensionless temperature $\dfrac{t - t_e}{qd/k}$;
ν, kinematic viscosity;
ρ, density;
τ, shear stress;
τ_w, shear stress at the wall;
η, dummy variable.

Suffixes

e, at the entrance position;
i, at one wall;
o, at the other wall;
1, fully developed;
2, in the entrance region;
m, bulk mean value;
n, eigenvalue number;
fd, fully developed.

INTRODUCTION

THE annular passage is an important heat transfer geometry in practice occurring in nuclear reactors and double pipe heat exchangers. At the present time, however, knowledge of the velocity and eddy diffusivity variations with turbulent flow in an annular passage is in an uncertain state. For turbulent flow in a passage formed by parallel walls infinite in extent, the flow is fairly well established and calculations can be carried out with some confidence. The parallel wall passage is a limiting case of the annulus where the radii are not greatly different and, for this reason, it is of direct interest to study this case.

The object of this article is to present solutions to the heat transfer problem with flow between parallel plates, i.e. to determine the variation of the Nusselt number with distance along the passage for a number of thermal boundary conditions which are of practical interest. An important case with flow in an annulus is that when the inner radius is heated and the outer radius is insulated and attention is focused in this article on the solution for the parallel passage with one side insulated. On the heated side two cases are considered, namely, uniform temperature and uniform heat input. These solutions are then extended to certain axial variations of temperature and heat input and in particular to the important case of a sinusoidal heat input variation together with a uniform heat input which occurs in reactor passages. Finally the solution is given for the case in which there are unequal uniform heating rates on both sides of the passage.

The fully developed situation for uniform but unequal heating rates has been studied by Barrow [1] who also obtained good experimental agreement. Barrow used similar assumptions to those made in this article in which the thermal entrance region is included.

The methods used are essentially the same as those given by Sparrow, Hallman and Siegel [2], by Sleicher and Tribus [3] and by Siegel and Sparrow [4, 5] all of whom gave solutions for flow in circular pipes. The main difference is that for fluids in circular pipes the temperature distribution is axisymmetrical. In such cases the heat flow in the vicinity of the pipe centre is small and the assumptions regarding the eddy diffusivity variation in this region have little effect on the result. With unsymmetrical boundary conditions however the variation used by these writers is unsatisfactory and a different approach must be adopted.

For axially varying heat input the solution follows the same procedure as a previous article by this writer on laminar flow in an annulus [6].

GENERAL EQUATIONS

The following assumptions are made:

1. Constant fluid physical properties.
2. Velocity profile fully developed at the duct entrance.
3. Heat transfer in the direction of flow is negligible.
4. Temperature changes due to dissipative effects are negligible.
5. Uniform temperature in the fluid at the duct entrance.

Using the concept of the eddy diffusivity of heat, the energy equation under these assumptions becomes

$$u \frac{\partial t}{\partial x} = \frac{\partial}{\partial y}\left[(\alpha + \epsilon_h)\frac{\partial t}{\partial y}\right]. \qquad (1)$$

Introducing the dimensionless variables

$$u^{+} = u\bigg/\sqrt{\left(\frac{\tau_w}{\rho}\right)}, \qquad y^{+} = \frac{y\sqrt{(\tau_w/\rho)}}{\nu}, \qquad x^{+} = \frac{x}{\;}$$

the equation becomes

$$u^{+}\frac{\partial t}{\partial x^{+}} = 4y_o^{+}\frac{\partial}{\partial y^{+}}\left[\left(\frac{\alpha + \epsilon_h}{\nu}\right)\frac{\partial t}{\partial y^{+}}\right]. \qquad (2)$$

The boundary conditions to be used will be discussed below.

VELOCITY AND EDDY DIFFUSIVITY VARIATION

Before equation (2) can be solved, the variations of u^{+} and $(\alpha + \epsilon_h)/\nu$ with y^{+} must be chosen. The solution can then be undertaken for a number of arbitrary values of y_o^{+}. Various proposals have been put forward for the $u^{+} - y^{+}$ relation and these are well described by Spalding

[7], who also proposed a new form in which the whole range is covered by a single equation.

The velocity profile and eddy diffusivity variation due to Deissler [8] have been used successfully to predict heat transfer in turbulent flow in tubes. The equations also give a close fit near the wall to the data of Corcoran *et al.* [9] who carried out extensive experiments in a parallel passage. In the central regions, however, Deissler's form of the law of the wall together with a linear shear stress variation leads to a low or negative value of the eddy diffusivity for momentum. In this region the equations were modified by assuming the eddy diffusivity constant and then deriving the velocity profile. This assumption has been used by other workers in obtaining results for the fully developed situation [1, 9, 10].

The value a^+ which is the position from which the eddy diffusivity is assumed constant is somewhat arbitrary. It appears from [9] that it may vary with Reynolds number but it lies in the range $0{\cdot}5y_o^+$ to $0{\cdot}7y_o^+$. From trial calculations made using different values of this parameter it appears that variations in this range affect the results only very slightly and a^+ was chosen at about $0{\cdot}5y_o^+$, which appears to fit the data of [9] reasonably well.

The assumption is also made that the eddy diffusivities for momentum and heat are equal despite the growing evidence that this is rather doubtful particularly at low Prandtl number. No reliable relation between the two quantities, however, has yet been established.

The system of equations used for describing the velocity and eddy diffusivity variation is as follows. The range of the equations is from 0 to $2y_o^+$ and they form symmetrical profiles about the centre line at y_o^+.

$$\left.\begin{aligned}
&0 < y^+ < 26,\\
&\frac{du^+}{dy^+} = \frac{1}{1 + n^2u^+y^+[1 - \exp(-n^2u^+y^+)]} \quad (n = 0{\cdot}0125)\\
&\frac{\alpha + \epsilon}{\nu} = \frac{1}{Pr} + n^2u^+y^+[1 - \exp(-n^2u^+y^+)]\\
&26 < y^+ < a^+,\\
&u^+ = 12{\cdot}8426 + \frac{1}{0{\cdot}36}\log\frac{y^+}{26}\\
&\frac{\alpha + \epsilon}{\nu} = \frac{1}{Pr} + 0{\cdot}36y^+\left(1 - \frac{y^+}{y_o^+}\right) - 1\\
&a^+ = (1/2)(y_o^+ + 26)\\
&a^+ < y^+ < y_o^+,\\
&u^+ = \frac{y^+}{\gamma} - \frac{y^{+2}}{2\gamma y_o^+} + B\\
&\text{where}\\
&\gamma = 0{\cdot}36a^+\left(1 - \frac{a^+}{y_o^+}\right)\\
&\text{and}\\
&B = 12{\cdot}8426 + \frac{1}{0{\cdot}36}\log\frac{a^+}{26} - \frac{1}{\gamma}\left(a^+ - \frac{a^{+2}}{2y_o^+}\right)\\
&\frac{\alpha + \epsilon}{\nu} = \frac{1}{Pr} + \gamma - 1.
\end{aligned}\right\} \quad (3)$$

It also follows from the definitions of u^+ and y^+ that

$$Re = 4\int_0^{y_o^+} u^+ \, dy^+. \quad (4)$$

From this equation the Reynolds number can be calculated from arbitrarily chosen y_o^+ and some values are given in Table 1.

Table 1. $Re - Y_o^+$ relation for turbulent flow between parallel plates

Re	Y_o^+
7096	126
14312	226
22036	326
42444	526
46992	626
73612	926
82748	1026
178568	2026
494576	5026
1063416	10026

SOLUTIONS OF THE GENERAL EQUATION

(1) *Uniform temperature on one wall, the other wall insulated*

The dimensionless temperature T is defined as

$$T = \frac{t - t_i}{t_e - t_i}. \quad (5)$$

In general, solutions consist of two parts, the fully developed temperature profile and an entrance solution which disappears at large x^+. It is clear that there is no fully developed profile in this case and the whole solution is given by the entrance solution.

Following the same reasoning as in [2] we obtain by the method of separation of variables

$$T = \sum_{n=1}^{\infty} C_n Y_n \exp\left(-8\lambda_n^2 x^+/Re\right) \quad (6)$$

where λ_n, Y_n are the eigenvalues and solutions of the Sturm–Liouville problem

$$\frac{d}{dy^+}\left[\left(\frac{\alpha+\epsilon}{\nu}\right)\frac{dY}{dy^+}\right] + \frac{2u^+\lambda^2}{Re \, . \, y_o^+} \cdot Y = 0 \quad (7)$$

$$Y = 0 \text{ at } y^+ = 0$$

$$\frac{dY}{dy^+} = 0 \text{ at } y^+ = 2y_o^+.$$

The constants C_n are given by

$$C_n = \frac{\int_0^{2y_o^+} u^+ Y_n \, . \, dy^+}{\int_0^{2y_o^+} u^+ Y_n^2 \, . \, dy^+}. \quad (8)$$

The solution must be carried out over the whole gap ($2y_o^+$) because of the unsymmetrical boundary conditions. The bulk mean temperature can be obtained from the solution and by using equation (7) expressed in the form

$$\frac{1}{Pr} \cdot Y'_{ni} = \frac{2\lambda_n^2}{Re \, . \, y_o^+} \int_0^{2y_o^+} u^+ \, . \, Y_n \, . \, dy^+ \quad (9)$$

which results in

$$T_m = \frac{y_o^+}{Pr} \sum_{n=1}^{\infty} \frac{C_n Y'_{ni}}{\lambda_n^2} \exp\left(-8\lambda_n^2 x^+/Re\right). \quad (10)$$

The Nusselt number then follows (using the heat-transfer coefficient based on the wall to bulk mean temperature difference)

$$Nu = \frac{4Pr \sum_{n=1}^{\infty} C_n Y'_{ni} \exp\left(-8\lambda_n^2 x^+/Re\right)}{\sum_{n=1}^{\infty} \frac{C_n Y'_{ni}}{\lambda_n^2} \exp\left(-8\lambda_n^2 x^+/Re\right)}. \quad (11)$$

These equations were solved using the Manchester University Mercury Computer. Table 2 gives the values of λ_n and $C_n Y'_{ni}$ for three Reynolds and three Prandtl numbers which may be used in (11) to derive the variation of Nusselt number with x^+.

(2) *Uniform heat input on one wall, the other insulated*

It is better in this case to use a dimensionless temperature θ defined from

$$\theta = \frac{t - t_e}{qd/k}. \quad (12)$$

The solution is obtained in two parts, θ_1 the fully developed temperature profile and θ_2 the entrance region profile which disappears as x^+ becomes large.

(a) *The fully developed profile*

A simple heat balance gives

$$\frac{d\theta_1}{dx^+} = \frac{2}{Re \, . \, Pr}$$

$$\frac{d}{dy^+}\left[\left(\frac{\alpha+\epsilon}{\nu}\right)\frac{d\theta_1}{dy^+}\right] = \frac{u^+}{2Re \, . \, Pr \, . \, y_o^+}. \quad (13)$$

The slope of θ_1 at the heat input side is given by

$$q = -k\left(\frac{dt}{dy}\right)_{y=0}$$

hence

$$\left(\frac{d\theta_1}{dy^+}\right)_{y^+=0} = -\frac{1}{4y_o^+} \quad (14)$$

and the solution is

$$\theta_1 = \frac{2}{Re \, . \, Pr} x^+ + G$$

where G is the fully developed profile.

(b) *The entrance region*

By the same methods as in the constant temperature case the solution is

$$\theta_2 = \sum_{n=1}^{\infty} C_n Y_n \exp\left(-8\lambda_n^2 x^+/Re\right) \quad (15)$$

where the λ_n and Y_n are eigenvalues and solutions of

Table 2. Eigenvalues and constants. Uniform temperature on one side, the other side insulated

n	$Pr = 0{\cdot}1$ λ_n	$C_n Y_{ni}'$	$Pr = 1{\cdot}0$ λ_n	$C_n Y_{ni}'$	$Pr = 10$ λ_n	$C_n Y_{ni}'$
$Re = 7096$						
1	4·57581	0·147757	2·55361	0·049787	1·38538	0·0151622
2	14·0447	0·105989	2·06643	0·0172367	7·81528	0·00144318
3	23·7382	0·081551	16·0894	0·009226	14·5916	0·0007255
4	33·4205	0·068169	22·8363	0·0074637	20·6899	0·0006865
5	42·9915	0·063112	29·1253	0·007433	25·9532	0·0008020
6	52·5218	0·060519	35·2911	0·0077921	30·9724	0·0009383
7	62·0815	0·057585	41·4927	0·0079426	36·0476	0·0010268
$Re = 73612$						
1	8·62619	0·0741824	6·08550	0·0390701	3·60242	0·0139752
2	27·6631	0·040843	23·4587	0·0101968	21·5889	0·0011034
3	47·3420	0·027441	42·0244	0·0052399	40·2887	0·0004807
4	66·5868	0·023149	52·6688	0·004032	57·7009	0·0003519
5	85·4110	0·022076	76·6308	0·0035303	74·3562	0·0002944
6	104·234	0·0211419	93·6654	0·0030976	91·1214	0·0002537
7	123·185	0·0199186	110·869	0·002764	108·022	0·0002309
$Re = 494576$						
1	16·8765	0·053976	12·7438	0·0318024	8·04710	0·0128536
2	57·8632	0·021774	52·9425	0·006566	49·9012	0·0009328
3	101·125	0·012881	95·9135	0·0032389	93·2850	0·0004012
4	142·559	0·010684	136·230	0·0024958	133·390	0·0002894
5	182·613	0·0101542	174·867	0·0021720	171·765	0·0002337
6	222·795	0·0096531	213·784	0·0018972	210·545	0·0001917
7	263·294	0·009272	253·016	0·0017133	249·663	0·0001646

$$\frac{\mathrm{d}}{\mathrm{d}y^+}\left[\left(\frac{\alpha+\epsilon}{\nu}\right)\frac{\mathrm{d}Y}{\mathrm{d}y^+}\right]+\frac{2\lambda^2 u^+ Y}{Re\,.\,y_o^+}=0 \quad (16)$$

$$\frac{\mathrm{d}Y}{\mathrm{d}y^+}=0 \quad \text{at } y^+=0 \quad \text{and } y^+=2y_o^+.$$

The symmetrical boundary conditions imply that the eigenfunctions will be either symmetrical or anti-symmetrical and a large saving in computing time can be made by using the half-range.

The constants C_n are given by

$$C_n=\frac{\displaystyle\int_0^{2y_o^+} u^+(-G)Y_n\,\mathrm{d}y^+}{\displaystyle\int_0^{2y_o^+} u^+Y_n^2\,\mathrm{d}y^+} \quad (17)$$

and the Nusselt number is easily shown to be

$$Nu=\frac{1}{G_i\left[1-\sum_{n=1}^{\infty}C_n\exp\left(-8\lambda_n^2x^+/Re\right)\right]} \quad (18)$$

where G_i is the difference between wall and mean temperatures on the heat input side for the fully developed profile.

Table 3 gives values of λ_n and C_n to enable equation (17) to be calculated.

(3) *Axial variation of temperature and heat input*

The above solutions may be extended in a straightforward manner to axial variations by the use of Duhamel's integral (see for example [4]). If $F(x^+)$ is the temperature variation at a particular value of y^+ caused by a unit step in either temperature or heat input at the wall then the response $Z(x^+)$ to an arbitrary variation $f(x^+)$ at the wall is given by

$$Z(x^+)=\int_0^{x^+}\frac{\partial f}{\partial x^+}F(x^+-z)\,\mathrm{d}z. \quad (19)$$

This integral has been calculated for a few cases of interest below.

Table 3. Eigenvalues and constants. Uniform heat flux on one side, the other side insulated

Re 7096						
	Pr 0·1		*Pr* 1·0		*Pr* 10	
G_i	0·2141052		0·7387098		0·02581698	
G_o	− 0·0768648		− 0·0148205		− 0·0018105	
n	λ_n	C_n	λ_n	C_n	λ_n	C_n
1	10·5851	0·453048	7·80291	0·263048	7·46092	0·090942
2	20·5723	0·140911	15·0006	0·104846	14·2449	0·048317
3	30·3202	0·076073	21·6133	0·078378	20·1805	0·053720
4	39·7985	0·050089	27·6359	0·066777	25·1755	0·065565
5	49·2543	0·035389	33·5781	0·054367	29·9135	0·069065
6	58·7887	0·026398	39·6470	0·042935	34·7616	0·064725
7	68·2729	0·020616	45·7495	0·033627	39·6840	0·057708
Re 73612						
	Pr 0·1		*Pr* 1·0		*Pr* 10	
G_i	0·0623753		0·0131940		0·0038308	
G_o	— 0·0181794		— 0·0020704		— 0·0002212	
n	λ_n	C_n	λ_n	C_n	λ_n	C_n
1	22·0430	0·376900	20·9064	0·205371	20·7885	0·072623
2	42·2838	0·133795	39·9055	0·080378	39·6525	0·029597
3	61·3268	0·080998	57·4954	0·054798	57·0367	0·021372
4	79·7950	0·055965	74·2859	0·042367	73·6425	0·0177233
5	98·4660	0·040181	91·2552	0·033870	90·3637	0·0154379
6	117·3824	0·030419	108·3859	0·028689	107·1940	0·0145074
7	136·2531	0·024265	125·3590	0·025626	123·7957	0·0146026
Re 494576						
	Pr 0·1		*Pr* 1·0		*Pr* 10	
G_i	0·0167821		0·0030369		0·0007734	
G_o	− 0·0037636		− 0·0003836		− 0·00003578	
n	λ_n	C_n	λ_n	C_n	λ_n	C_n
1	48·7477	0·294157	48·2497	0·167790	48·1997	0·062193
2	93·1388	0·112429	92·0672	0·066072	91·9585	0·026276
3	134·036	0·073979	132·2351	0·045081	132·0501	0·018122
4	173·328	0·054151	170·6821	0·034155	170·4069	0·013909
5	213·136	0·040350	209·593	0·026287	209·2205	0·010878
6	253·358	0·031384	248·853	0·021096	248·3750	0·008909
7	293·274	0·025240	287·759	0·017469	287·1742	0·007569

(a) *Linear rise of wall temperature*

For this case, assuming the other wall insulated, then $f(x^+) = kx^+$ and the Nusselt number is given by

$$Nu = \frac{4Pr \cdot \sum_{n=1}^{\infty} \frac{C_n Y'_{ni}}{8\lambda_n^2/Re}\,[1 - \exp(-8\lambda_n^2 x^+/Re)]}{\sum_{n=1}^{\infty} \frac{C_n Y'_{ni}}{8\lambda_n^4/Re}\,[1 - \exp(-8\lambda_n^2 x^+/Re)]}. \tag{20}$$

The λ_n and $C_n Y'_{ni}$ are here taken from Table 2 for the uniform wall temperature case.

(b) *Linear variation of heat input*

In this case we assume an imposed heat transfer on one side only which varies linearly with distance along the passage, again $f(x^+) = kx^+$ and the result is

$$Nu = \frac{x^+}{G_i\left\{x^+ - \sum_{n=1}^{\infty} \frac{C_n}{8\lambda_n^2/Re}\,[1 - \exp(-8\lambda_n^2 x^+/Re)]\right\}}. \tag{21}$$

In this expression the λ_n and C_n are for the uniform heat input case and are taken from Table 3.

(c) *Sinusoidal heat input variation superimposed on a uniform heat input*

This form of axial variation occurs from the fuel rod in a reactor channel. Here

$$f(x^+) = 1 + A \sin \frac{\pi x^+}{x_1^+}$$

where x_1^+ is the passage overall length and A is the ratio of the maximum component of heat flux due to sine variation to the uniform heat flux component.

The Nusselt number becomes

$$Nu = \frac{\frac{1}{G_i}\left(1 + A \sin \frac{\pi x^+}{x_1^+}\right)}{1 - \sum_{n=1}^{\infty} C_n \exp(-8\lambda_n^2 x^+/Re) + A \sin \frac{\pi x^+}{x_1^+} - \frac{A\pi}{x_1^+} \sum_{n=1}^{\infty} \frac{C_n}{\frac{\pi^2}{x_1^{+2}} + \left(\frac{8\lambda_n^2}{Re}\right)^2} \times \left[\frac{8\lambda_n^2}{Re} \cos \frac{\pi x^+}{x_1^+} + \frac{\pi}{x_1^+} \sin \frac{\pi x^+}{x_1^+} - \frac{8\lambda_n^2}{Re} \exp(-8\lambda_n^2 x^+/Re)\right]}. \tag{22}$$

Again, in this expression, the λ_n, C_n are to be taken from Table 3.

RESULTS FOR UNIFORM AND AXIALLY VARYING BOUNDARY CONDITIONS

Before discussing the results of the calculations described above it is worth bearing in mind that for the cases of prescribed wall heat flux the problem would be to determine the wall temperature variation with x^+. In fact, expressions can be derived for the wall temperature which are rather simpler than the above Nusselt number expressions. However, it is convenient for comparison purposes to give the results in terms of Nusselt numbers and it is not difficult to recalculate the wall temperature variation.

The variation of Nusselt number with x^+ is plotted on Figs. 1, 2 and 3 for three different Prandtl numbers. On each figure the four boundary conditions of uniform temperature and heat input and linear rise of temperature and heat input (referred to as "ramp") are shown.

The influence of the type of boundary condition is very small in the fully developed situation, a conclusion which was also obtained by Siegel and Sparrow [5] for round tubes. The thermal entrance length itself does not change much over the range of variables chosen but it is rather longer for these unsymmetrical cases than those given for round tubes.

Fig. 4 summarizes the fully developed values, and, for comparison, the results of Barrow [1] are included. All previous experimental correlations have been concerned with round tubes and close agreement with such work would not be expected. Accepted correlations for round tubes are for $0{\cdot}7 < Pr < 100$, $Nu_{fd} = 0{\cdot}024\, Re^{0{\cdot}8}\, Pr^{0{\cdot}4}$ from Dittus and Boelter given in Eckert and

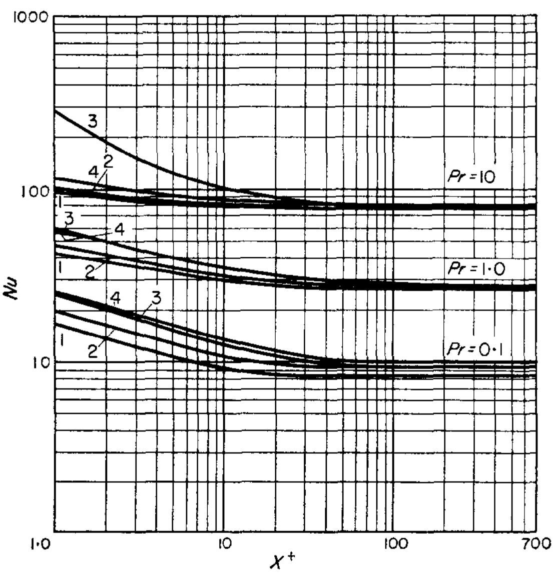

FIG. 1. Nu vs. x^+ curves for various forms of thermal boundary condition on one side, the other side insulated (Re 7096).

1. Const. T.
2. Const. q.
3. Ramp T.
4. Ramp q.

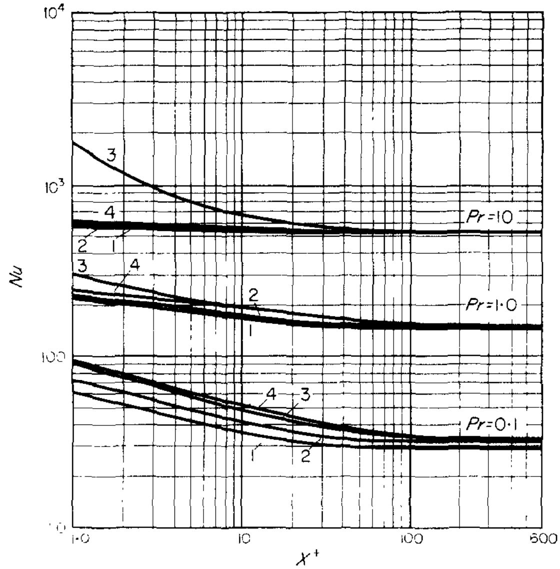

FIG. 2. *Nu* vs. x^+ curves for various forms of thermal boundary condition on one side, the other side insulated (*Re* 73612).

1. Const. *T*.
2. Const. *q*.
3. Ramp *T*.
4. Ramp *q*.

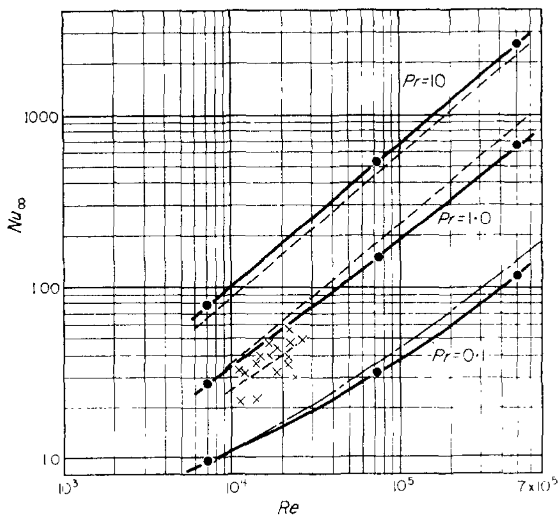

FIG. 4. Fully developed Nusselt numbers for uniform heat flux on one side.
(Barrow [1])

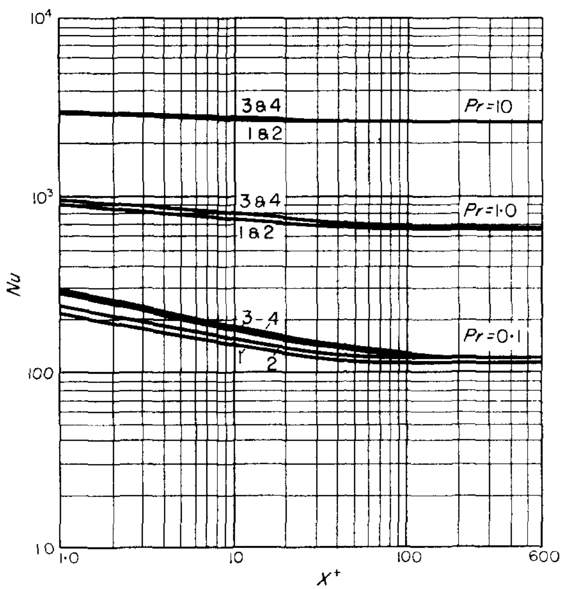

FIG. 3. *Nu* vs. x^+ curves for various forms of thermal boundary condition on one side, the other side insulated (*Re* 494576).

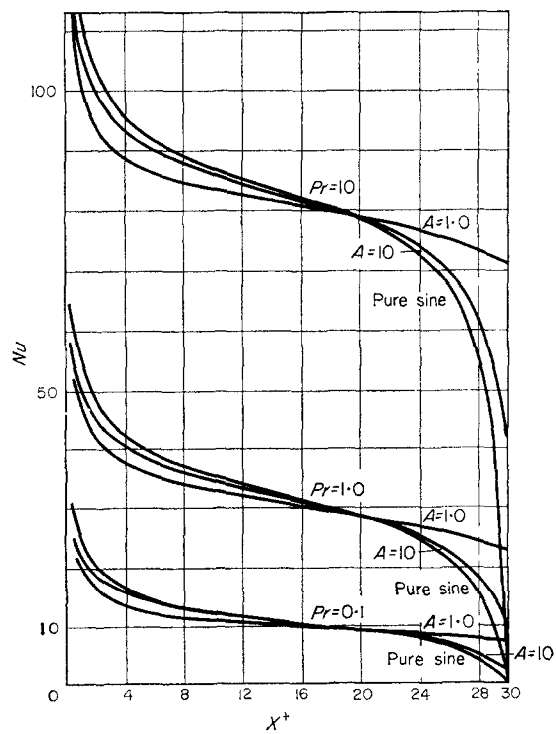

FIG. 5. Nusselt number variation with distance for a half sine-wave heat flux superimposed on a uniform heat flux, one side being insulated (*Re* 7096, $X_1{}^+ = 30$).

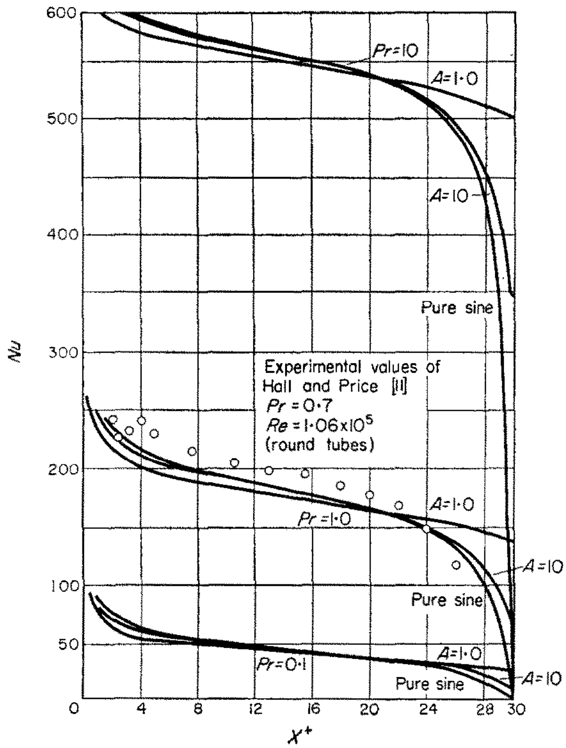

FIG. 6. Nusselt number variation with distance for a half sine-wave heat flux superimposed on a uniform heat flux, one side being insulated (Re 73612, $X_1^+ = 30$).

Drake [12] and for $Pr \leqslant 0{\cdot}1$, $Nu_{fd} = 5{\cdot}0 + 0{\cdot}025\,(Re\,.\,Pr)^{0{\cdot}8}$ from Seban and Shimazaki [13].

These lines are added to Fig. 4 and confirm that the calculated values are of the same order as those for round tubes.

Figs. 5, 6 show results for a few cases of equation (22); for an arbitrarily chosen value of $x_1^+ = 30$. Equation (22) has two extreme cases, namely uniform heat input ($A = 0$) and a pure sine wave heat input ($A = \infty$). The curves show that in passages of this order of length 30 × equivalent diameter) the Nusselt number varies considerably. It is of interest to show the experimental values of Hall and Price [11], although these are for a circular tube and at a rather different Reynolds number, which were obtained on a length of 30d and with a series of step heat inputs adjusted to conform to a sine wave.

(4) *Unequal uniform heat inputs on each side*

Results for this case can be easily derived from those of case 2 (uniform heat input on one side, the other adiabatic) by straightforward superposition of temperature profiles due to a *unit* heat input.

As the parallel plate case is entirely symmetrical, the only additional value requiring calculation is G_o which is the fully developed dimensionless temperature on the adiabatic side in case 2. This is also quoted in Table 3.

The Nusselt number on the side i can be written

$$Nu_i = \frac{1}{[G_i + \sum_{n=1}^{\infty} C_n Y_{ni} \exp(-8\lambda_n^2 x^+/Re)] + \frac{q_o}{q_i}[G_o + \sum_{n=1}^{\infty} C_n Y_{no} \exp(-8\lambda_n^2 x^+/Re)]}. \tag{23}$$

In this calculation Y_{ni} is equal to $-G_i$ but these are not equal to G_o. However, due to the anti-symmetrical nature of the eigenfunctions, Y_{ni} is numerically equal to Y_{no}, but the Y_{no} alternate in sign. The above equation can thus be written

$$Nu_i = \frac{1}{G_i\,[1 - \sum_1^{\infty} C_n \exp(-8\lambda_n^2 x^+/Re)] + \frac{q_o}{q_i}[G_o - G_i \sum_1^{\infty} C_n (-1)^n \exp(-8\lambda_n^2 x^+/Re)]} \tag{24}$$

and for the side 0

$$Nu_o = \frac{1}{G_i\,[1 - \sum_1^{\infty} C_n \exp(-8\lambda_n^2 x^+/Re)] + \frac{q_i}{q_o}[G_i - G_i \sum_1^{\infty} C_n (-1)^n \exp(-8\lambda_n^2 x^+/Re)]}. \tag{25}$$

These expressions may sometimes produce an unusual behaviour in the Nusselt number. For example, Fig. 7 shows the variation of Nu_o for $q_i/q_o = -10$ and -7. The Nusselt number in each case passes through a singular point and becomes negative. This singular value corresponds to the point at which the bulk temperature passes the wall temperature and hence the temperature difference changes sign.

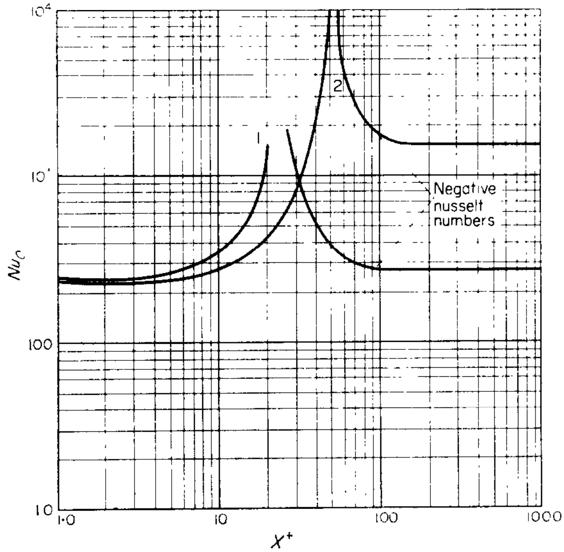

FIG. 7. Nusselt number variation on the side o for unequal heat fluxes on each side of the passage (*Re* 73612, $Pr = 1{\cdot}0$).

1. $q_i/q_o = -10$. 2. $q_i/q_o = -7$.

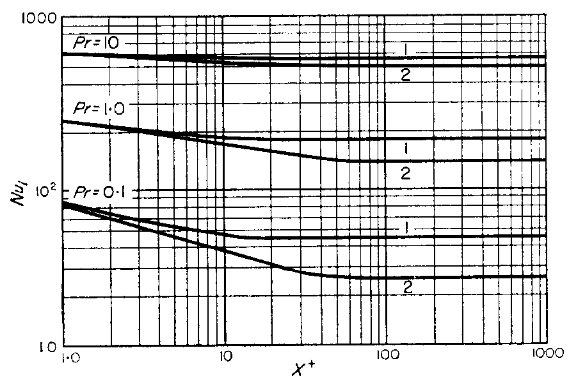

FIG. 8. Nusselt number variation for equal heat inputs on each side and for equal heat input and output on each side (*Re* 73612).

1. $q_i/q_o = -1$. 2. $q_i/q_o = +1$.

Two special cases of interest are when $q_i/q_o = +1$ and -1 corresponding to equal heat input and output and equal heat inputs respectively. It is seen on Fig. 8 that the Nusselt number may be appreciably affected by the unsymmetrical boundary condition.

Fig. 9 shows the effect of different values of q_i/q_o on the fully developed Nusselt number. The ordinate is the ratio of $(Nu_\infty)_o$ for a particular value of q_i/q_o to $(Nu_\infty)_o$ for heating on one side only. Hence all the curves pass through the point (1·0, 0).

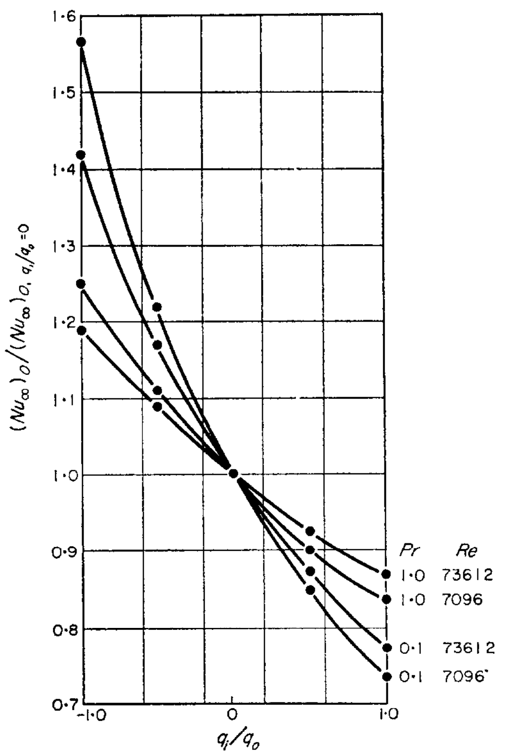

FIG. 9. Effect of unsymmetrical heat fluxes on fully developed Nusselt numbers.

In a similar fashion Fig. 10 shows the effect of the same parameter on the thermal entry length. This length $x^+_{fd\,o}$ is taken as that at which the Nusselt number is within 5 per cent of the value at infinity. The diagram shows that the unsymmetrical boundary condition greatly affects the thermal entrance length.

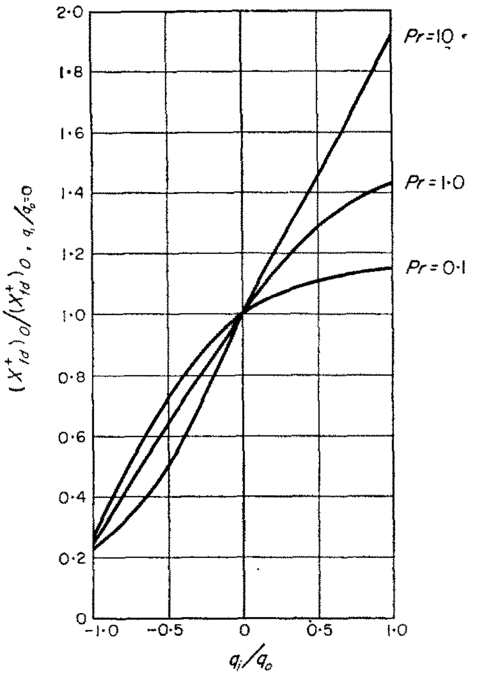

FIG. 10. Effect of unsymmetrical heat fluxes on thermal entry length.

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Résumé—On présente ici la solution du problème de la transmission de chaleur dans un écoulement turbulent entre deux plaques parallèles chauffées sur une seule face. Cette configuration a une signification pratique quand on la considère comme le cas limite de l'anneau à noyau chauffé. Les variations de vitesse et de diffusivité turbulente données par Deissler sont utilisées, mais sont modifiées dans la région centrale du passage en supposant une diffusivité turbulente constante. Dans la résolution de l'équation de l'énergie, les diffusivités turbulentes de chaleur et du quantité de mouvement sont considérées comme étant égales. Les valeurs propres et les fonctions nécessaires au calcul de la variation du nombre de Nusselt en fonction de la distance au long des passages sont tabulées pour trois nombres de Reynolds et trois nombres de Prandtl.

A partir de ces deux solutions fondamentales correspondant à une température constante et à un flux de chaleur constant, on étend l'analyse au cas de certaines variations axiales intéressantes en pratique et, finalement, par superposition on obtient la solution pour des flux de chaleur constants mais différents pour chacun des côtés du passage.

Zusammenfassung—Für das Wärmeübergangsproblem mit turbulenter Strömung zwischen parallelen Platten bei Heizung nur von einer Seite ist eine Lösung angegeben. Die Anordnung hat praktische Bedeutung als Grenzfall eines Ringraumes mit beheiztem Kern. Nach Deissler werden die Änderungen der Geschwindigkeit und des turbulenten Austausches herangezogen; für den Kernbereich sind sie jedoch durch die Annahme konstanter Austauschgrösse abgewandelt. Für die Lösung der Energiegleichung sind die Austauschgrössen für Wärme und Impuls als gleich gross angenommen. Die Eigenwerte und Funktionen, die zur Berechnung der Änderung der Nusseltzahl längs der Platten dienen, sind für drei Reynolds- und drei Prandtlzahlen tabelliert. Aus den beiden Grundlösungen für konstante Temperatur und konstanten Wärmefluss ist die Analyse auf bestimmte achsiale Änderungen, soweit sie von praktischem Interesse sind, ausgedehnt. Durch Superposition erhät man das Ergebnis für konstanten, jedoch für jede Wand verschiedenen Wärmestrom.

Аннотация—Представлено решение задачи о переносе тепла при турбулентном течении между параллельными пластинами при одностороннем нагревании. Такая конфигурация имеет определенное практическое значение как предельный случай кольцевого канала с нагреваемым внутренним стержнем. Скорость и коэффициент турбулентной диффузии берутся по Дайсслеру, однако в центральной области канала коэффициент турбулентной диффузии принят постоянным. При решении уравнения энергии значения коэффициентов турбулентной диффузии тепла и количества движения берутся равными. Собственные значения и функции, необходимые для расчёта изменения числа Нуссельта с изменением расстояния вдоль каналов, табулированы для трех значений чисел Рейнольдса и Прандтля.

Два основных решения для постоянной температуры и однородного теплового потока распространены на некоторые случаи аксиальных изменений, представляющих практический интерес. И наконец, используя суперпозицию, получен результат для неравных однородных тепловых потоков на каждой стороне канала.